Optimal and Constrained-Optimal Control of a Flexible Launch Vehicle

M. C. MAKI* AND J. VAN DE VEGTE†
Department of Mechanical Engineering, University of Toronto, Toronto, Canada

The constant gain feedback control of a flexible launch vehicle is considered for the case when the feedback signals are obtained from only three sensors. Sensor feedback gains as well as sensor positions on the vehicle are computed to optimize the response. Performance is very close to that found by optimal control with complete state feedback. Efficient numerical techniques, based on transformation of the single-input system to control canonical form, are used to solve the optimal and constrained-optimal problems.

Nomenclature

α = angle of attack
β = engine gimbal angle
ϕ = attitude angle
η_i = ith flexure mode
V = vehicle velocity
x = distance from vehicle tail

Introduction

IN general, an uncontrolled flexible launch vehicle is an unstable system with lightly damped modes of transverse vibration. Optimal control is in general too complex for implementation because it requires a number of measurements equal to the number of state variables. Therefore both Rynaski[1] and Fisher[2] have considered suboptimal design constrained to fewer measurements.

Rynaski assumes three sensors at given positions on a rocket described by an eighth-order state-space model, including two bending modes. A reasonable approximation to the response of a chosen model is obtained by the use of filters to modify the sensor outputs. Fisher assumes six sensors at given positions on a rocket described by a ninth-order state-space model, including three bending modes. With six sensors, feedback gains quite close to the optimal gains with complete state feedback are obtained, but the method does not guarantee stability.

In this paper, a control to minimize a performance index is designed using only three sensors and constant feedback gains. Not only the feedback gains but also the sensor positions required to minimize the cost functional are determined by parameter optimization.

Rocket Dynamics and Sensor Output Equations

Fisher's numerical model[2] was chosen for use in this study. This model is for the pitch plane motion of a five-engine flexible rocket booster. The model is linear and time-invariant, and includes three bending modes. Engine dynamics and the effect of angle of attack of the rocket on the bending vibrations are neglected.

The equations for rocket motion, obtained from geometry, from the dynamic equilibrium of the rocket, and from consideration of the rocket as a free-free beam to derive the bending mode equations, are transformed by Fisher into the ninth-order state-space model given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \qquad (1)‡$$

Received October 18, 1971; revision received January 20, 1972. This work was supported by the National Research Council of Canada.
Index category: Launch Vehicle and Controlled Missile Dynamics and Control.

* Graduate Student.
† Associate Professor.
‡ Symbols in boldface type indicate vectors or matrices.

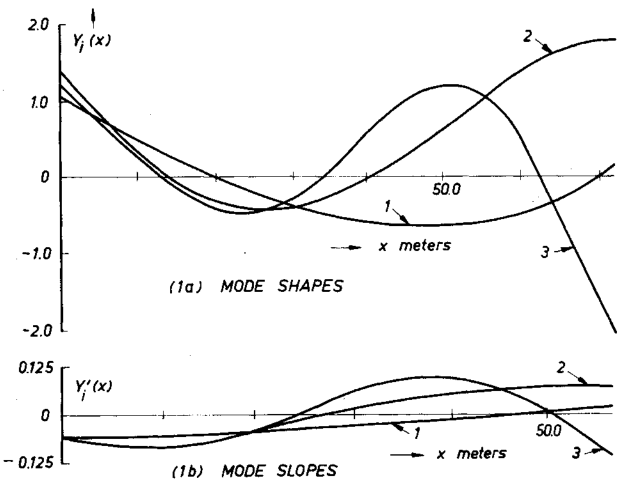

Fig. 1 a) Mode shape functions; b) mode slope functions.

where matrix $\mathbf{A}$ and vector $\mathbf{b}$ are given numerically in Table 1.

In this equation the scalar input variable u is the engine gimbal angle β, $u = \beta$, and the components x_1 to x_9 of the state vector $\mathbf{x}$ are defined in terms of the original variables as follows, where the prime denotes the transpose and where α is the angle of attack, ϕ the pitch angle, and η_i the ith bending mode.

$$\begin{aligned}\mathbf{x}' &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \\ &= (\phi, \dot{\phi}, \alpha, \eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \eta_3, \dot{\eta}_3)\end{aligned} \qquad (2)$$

Fisher expresses the outputs of angular position sensors, angular rate sensors, and accelerometers, respectively, in terms of linear combinations of the state variables as follows:

$$\begin{aligned}s_p(x) &= \sum_{j=1}^{9} z_{pj}(x)x_j; \qquad s_r(x) = \sum_{j=1}^{9} z_{rj}(x)x_j \\ s_a(x) &= \sum_{j=1}^{9} z_{aj}(x)x_j\end{aligned} \qquad (3)$$

The coefficients $z(x)$ in these equations are functions of distance x of the sensor to the tail of the rocket, and are given in Table 2, together with the bending mode shape functions on which they depend. To interpret sensor positions, Fig. 1 shows plots of mode shapes and mode shape derivatives.

Equations (3) can be written in the usual form

$$\mathbf{y} = \mathbf{C}\mathbf{x} \qquad (4)$$

where vector $\mathbf{y}$ represents the sensor outputs, and the elements of measurement matrix $\mathbf{C}$ can be computed for a given distribution of sensors.

Performance Criterion

The suboptimal control must be designed to minimize the quadratic performance index

Table 1 Matrices for state-space model

$$
A = \begin{bmatrix}
0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.2165 & -0.0356 & 0 & -0.299 & 0 & -0.0270 & 0 \\
-0.0458 & 1.00 & -0.0133 & 0.0004 & 0 & 0.0006 & 0 & 0.0007 & 0 \\
0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -29.81 & -0.0546 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -169.0 & -0.1300 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -334.3 & -0.1828
\end{bmatrix}
$$

$$
b' = [\; 0 \quad -1.138 \quad -0.348 \quad 0 \quad 29.56 \quad 0 \quad 47.25 \quad 0 \quad 16.40 \;]
$$

$$
Q = \begin{bmatrix}
0.10 & 0 & -0.10 & & & & & & \\
0 & 0.05 & 0 & & & & & 0 & \\
-0.10 & 0 & 0.50 & & & & & & \\
 & & & 0.0001 & & & & & \\
 & & & & 0.003 & & & & \\
 & & & & & 0.00008 & & & \\
 & & & 0 & & & 0.0137 & & \\
 & & & & & & & 0.00019 & \\
 & & & & & & & & 0.0621
\end{bmatrix}
$$

$$
J = \frac{1}{2}\int_0^\infty (\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})\, dt \tag{5}
$$

where for notational convenience the scalar u has been written as a vector.

The weighting matrix $\mathbf{R} = \mathbf{I}$, the identity matrix, and the numerical matrix $\mathbf{Q}$ is given in Table 1. This $\mathbf{Q}$ was found in the usual iterative fashion by examining optimal trajectories resulting from certain choices of its elements, and applying the physical criteria for optimality. The lateral drift velocity of the rocket is $V(\phi - \alpha)$, where V is vehicle velocity. The two nonzero off-diagonal terms of Q help to provide weighting on $\phi - \alpha$, to restrict drift.

Bending loads are related to engine gimbal angle β and angle of attack α. It is known that generally considerably larger values of α than of β can be tolerated. Hence the diagonal element of Q which corresponds to x_3 must be smaller than unity. The weighting element corresponding to x_2 serves to restrict pitch velocity.

The remaining diagonal elements of $\mathbf{Q}$ have been chosen to weight the energies in the bending modes equally. It can readily be shown that the kinetic energy T and potential energy U in the bending modes are given by

$$
T = \frac{1}{2}\sum_{i=1}^{3} M_i \dot{\eta}_i^2
$$

Table 2 Coefficients for sensor output equations

Sensor equation coefficients

$z_{p1} = 1, z_{r1} = 0, z_{a1} = 0, z_{p2} = 0, z_{r2} = 1, z_{a2} = 0$

$z_{p3} = 0,\ z_{r3} = 0,\ z_{a3} = -0.060 + 0.2165x,\ z_{p4} = Y_1'(x)$

$z_{r4} = 0,\ z_{a4} = 0.826 - 0.0356x - 29.81\,Y_1(x) + 20.89\,Y_1'(x)$

$z_{p5} = 0,\ z_{r5} = Y_1'(x),\ z_{a5} = -0.0545\,Y_1(x),\ z_{p6} = Y_2'(x)$

$z_{r6} = 0,\ z_{a6} = 0.575 - 0.0299x - 169\,Y_2(x) + 20.89\,Y_2'(x)$

$z_{p7} = 0,\ z_{r7} = Y_2'(x),\ z_{a7} = -0.130\,Y_2(x),\ z_{p8} = Y_3'(x)$

$z_{r8} = 0,\ z_{a8} = 0.445 - 0.0270x - 334.3\,Y_3(x) + 20.89\,Y_3'(x)$

$z_{p9} = 0,\ z_{r9} = Y_3'(x),\ z_{a9} = -0.1828\,Y_3(x)$

Mode shape functions

$Y_1(x) = 1.045 - 0.0632x + 0.5915 \times 10^{-3}x^2 + 0.3273 \times 10^{-5}x^3 - 0.2429 \times 10^{-6}x^4 + 0.4599 \times 10^{-8}x^5 - 0.2156 \times 10^{-10}x^6$

$Y_2(x) = 1.349 - 0.0779x - 0.5233 \times 10^{-2}x^2 + 0.4182 \times 10^{-3}x^3 - 0.1115 \times 10^{-4} + 0.1575 \times 10^{-6}x^5 - 0.1167 \times 10^{-8}x^6 + 0.3451 \times 10^{-11}x^7$

$Y_3(x) = 1.079 - 0.0598x - 0.1566 \times 10^{-2}x^2 - 0.1956 \times 10^{-3}x^3 + 0.1931 \times 10^{-4}x^4 - 0.4760 \times 10^{-6}x^5 + 0.4577 \times 10^{-8}x^6 - 0.1527 \times 10^{-10}x^7$

Mode slope functions

$Y_1'(x) = dY_1(x)/dx,\ Y_2'(x) = dY_2(x)/dx,\ Y_3'(x) = dY_3(x)/dx$

and

$$
U = \frac{1}{2}\sum_{i=1}^{3} \omega_i^2 M_i \eta_i^2
$$

where M_i and ω_i are the generalized mass and the undamped natural frequency in the ith mode. Equal weighting therefore requires that the ratio of the last six diagonal elements of $\mathbf{Q}$ be

$$
1 : \omega_1^2 : M_2/M_1 : M_2\omega_2^2/M_1\omega_1^2 : M_3/M_1 : M_3\omega_3^2/M_1\omega_1^2
$$

Optimal and Quadratic Suboptimal Control

For the state model (1) with output Eq. (4), it is desired to find the constant gain feedback

$$
u = \mathbf{k}\mathbf{y} \tag{6}
$$

which minimizes performance index (5).

It is well known,[3] that with complete state feedback the optimal feedback gain vector is given by

$$
\mathbf{k} = -\mathbf{R}^{-1}\mathbf{b}'\mathbf{K}\mathbf{C}^{-1} \tag{7}
$$

where $\mathbf{K}$ is the unique positive definite symmetric solution of the matrix Riccati equation

$$
\mathbf{A}'\mathbf{K} + \mathbf{K}\mathbf{A} - \mathbf{K}\mathbf{b}\mathbf{R}^{-1}\mathbf{b}'\mathbf{K} + \mathbf{Q} = \mathbf{0} \tag{8}
$$

The problem of incomplete state feedback, when $\mathbf{C}^{-1}$ does not exist, has recently been considered by several investigators.[3-6]

From Eqs. (1, 4, and 6) $\dot{\mathbf{x}} = [\mathbf{A} + \mathbf{bkC}]\mathbf{x}$, so that $\mathbf{x}(t) = \exp\{[\mathbf{A} + \mathbf{bkC}]t\}\mathbf{x}(0) \equiv \gamma(t,0)\mathbf{x}(0)$. Substitution into (5) yields

$$
\begin{aligned}
J &= \frac{1}{2}\mathbf{x}'(0)\left[\int_0^\infty \gamma'(t,0)(\mathbf{Q} + \mathbf{C}'\mathbf{k}'\mathbf{R}\mathbf{k}\mathbf{C})\gamma(t,0)\,dt\right]\mathbf{x}(0) \\
&\equiv \tfrac{1}{2}\mathbf{x}'(0)\mathbf{V}(\mathbf{k})\mathbf{x}(0)
\end{aligned} \tag{9}
$$

$\mathbf{V}(\mathbf{k})$ and $\gamma(t,0)$ are defined by these equations. To remove the undesirable dependence on initial condition, $\mathbf{x}(0)$ is assumed to be a random variable uniformly distributed on the surface of an n-dimensional unit sphere, where n is the order of the system. Then n times the expected value of J is independent of $\mathbf{x}(0)$ and given by

$$
\hat{J} = \tfrac{1}{2}\,\text{trace}\,[\mathbf{V}(\mathbf{k})] \tag{10}
$$

This new criterion retains many of the properties of (9). It gives a design which is optimal in an average sense, and also provides a simple upper bound on worst case performance.

It can be shown that the suboptimal control problem can also be stated as follows. Find vector $\mathbf{k}$ to minimize $\hat{J} = \frac{1}{2}$ trace $[\mathbf{V}(\mathbf{k})]$ where $\mathbf{V}$ is the symmetric positive definite solution of the Liapunov equation

$$
(\mathbf{A} + \mathbf{bkC})'\mathbf{V} + \mathbf{V}(\mathbf{A} + \mathbf{bkC}) + \mathbf{C}'\mathbf{k}'\mathbf{R}\mathbf{k}\mathbf{C} + \mathbf{Q} = \mathbf{0} \tag{11}
$$

A positive definite solution exists if a stable matrix $(\mathbf{A} + \mathbf{bkC})$ can be found.

Necessary conditions for minimization of (10) have been derived by Levine and Athans as a set of three coupled equations, two of which are Liapunov equations. A recursive method suggested for their solution does not always converge, as has been noted by Davison. Axsäter[5] considers the conditions for minimization and solution of the general time-variable problem.

An alternate approach to solution of the optimization problem (11) is based on parameter optimization and has been applied by Davison and Rau, using Rosenbrock's method. This approach has been adopted in this paper to optimize sensor feedback gains and sensor positions. It is to be noted that measurement matrix **C** depends on sensor positions.

Numerical Solution

Kleinman[7] has shown that the recursive solution of a particular Liapunov equation converges to the solution of the Riccati equation, required in the optimal control problem. Since for the suboptimal control Liapunov equation (11) must be solved at each stage of the parameter optimization process, efficient solution of the Liapunov equation is the key to efficient computation for both the optimal and suboptimal problems. The numerical integration process proposed by Davison and Man[8] is found to be too slow for the lightly damped system under study.

The method proposed here applies to single-input systems and exploits the structural advantages obtained by transforming such systems to the control canonical form. It allows solution of the Liapunov equation independent of the system damping characteristics.

The control canonical form of (1) is

$$\dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{g}u, \qquad \mathbf{z} = \mathbf{H}\mathbf{x} \tag{12}$$

where

$$\mathbf{F} = \mathbf{H}\mathbf{A}\mathbf{H}^{-1} = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -a_1 & \cdots & \cdots & \cdots & -a_n \end{bmatrix}, \qquad \mathbf{g} = \mathbf{H}\mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

in which the a_i are the coefficients of the system characteristic polynomial $s^n + a_n s^{n-1} + \ldots + a_2 s + a_1$.

By Krylov's method,[9] the coefficients a_i are given by

$$[a_1, \ldots, a_n]' = -[\mathbf{b}, \mathbf{A}\mathbf{b}, \ldots, \mathbf{A}^{n-1}\mathbf{b}]^{-1}(\mathbf{A}^n\mathbf{b}) \tag{13}$$

and Kalman[10] has given H^{-1}, which in our case becomes as shown below, so that it can be found by simple matrix multiplication

$$\mathbf{H}^{-1} = [\mathbf{b}, \mathbf{A}\mathbf{b}, \ldots, \mathbf{A}^{n-1}\mathbf{b}]\,[\mathbf{g}, \mathbf{F}\mathbf{g}, \ldots, \mathbf{F}^{n-1}\mathbf{g}]^{-1}$$

$$= [\mathbf{b}, \ldots, \mathbf{A}^{n-1}\mathbf{b}] \begin{bmatrix} a_2 & \cdots & \cdots & a_n & 1 \\ \vdots & & & 1 & \\ \vdots & & & & \\ a_n & 1 & & 0 & \\ 1 & & & & \end{bmatrix} \tag{14}$$

Considering first the case of optimal control, the transformed performance index, optimal control, and Riccati equation are

$$J = \frac{1}{2}\int_0^\infty (\mathbf{z}'\mathbf{H}^{-1\prime}\mathbf{Q}\mathbf{H}^{-1}\mathbf{z} + u'Ru)\,dt$$

$$u_{\text{opt}} = -R^{-1}\mathbf{g}'\mathbf{K}\mathbf{z} \tag{15}$$

$$\mathbf{F}'\mathbf{K} + \mathbf{K}\mathbf{F} - \mathbf{K}\mathbf{g}R^{-1}\mathbf{g}'\mathbf{K} + \mathbf{H}^{-1\prime}\mathbf{Q}\mathbf{H}^{-1} = \mathbf{0}$$

Kleinman[7] has shown that this matrix Riccati equation can be solved by recursive solution of a Liapunov equation, which at the ith stage of iteration is

$$\mathbf{F}_i'\mathbf{K}_i + \mathbf{K}_i\mathbf{F}_i + \mathbf{L}_i'R\mathbf{L}_i + \mathbf{H}^{-1\prime}\mathbf{Q}\mathbf{H}^{-1} = \mathbf{0} \tag{16}$$

where

$$\mathbf{L}_i = R^{-1}\mathbf{g}'\mathbf{K}_{i-1}, \qquad \mathbf{F}_i = \mathbf{F} - \mathbf{g}\mathbf{L}_i$$

If $\mathbf{L}_0$ is chosen so that $\mathbf{F}_0$ is stable, then $\mathbf{K} \leqq \mathbf{K}_{i+1} \leqq \mathbf{K}_i$ and

$$\lim_{i\to\infty} \mathbf{K}_i = \mathbf{K}$$

Because $\mathbf{F}_i$ is a companion matrix, i.e., of the same form as **F** in (12), Eq. (16) may be solved efficiently using an algorithm of Molinari.[11] With this algorithm Eq. (16) can be re-ordered into the form $\mathbf{D}\mathbf{w} = \mathbf{e}$, a set of $n = 9$ linear algebraic equations, which can be solved directly by Gaussian elimination. **D** is a Hurwitz matrix, and vector **w** involves only the final row of $\mathbf{K}_i$, the only terms required to determine the feedback in (15).

The pole-placement method is used to choose $\mathbf{L}_0$ such that $\mathbf{F}_0$ is stable. **F** is the companion matrix of the open loop characteristic polynomial $s^n + a_n s^{n-1} + \ldots + a_2 s + a_1$. Let **Z** be the companion matrix of some stable characteristic polynomial arbitrarily chosen as $(s+2)^9$ in this work. Then it is easily shown by substitution into $\mathbf{F} - \mathbf{g}\mathbf{L}_0$ that the choice $\mathbf{L}_0 = \mathbf{g}'(\mathbf{F} - \mathbf{Z})$ will realize this stable polynomial.

Usually less than 30 Liapunov equation solutions are required for convergence to **K**. The computer operation count, in terms of multiplications for an nth-order system, is about $\frac{16}{3}n^3$ for the canonical transformation and $\frac{1}{3}n^3$ for each Liapunov equation solution, so that the total operation count is about

$$(\tfrac{16}{3} + 30 \times \tfrac{1}{3})n^3 \simeq 15n^3$$

For the suboptimal control, Liapunov Eq. (11) is transformed to the control canonical form below, and is solved in the manner discussed at each stage of the parameter optimization process. With $\hat{\mathbf{C}} = \mathbf{C}\mathbf{H}^{-1}$

$$(\mathbf{F} + \mathbf{g}\mathbf{k}\hat{\mathbf{C}})'\mathbf{V} + \mathbf{V}(\mathbf{F} + \mathbf{g}\mathbf{k}\hat{\mathbf{C}}) + \hat{\mathbf{C}}'\mathbf{k}'R\mathbf{k}\hat{\mathbf{C}} + \mathbf{H}^{-1\prime}\mathbf{Q}\mathbf{H}^{-1} = \mathbf{0} \tag{17}$$

The closed loop system matrix $(\mathbf{F} + \mathbf{g}\mathbf{k}\hat{\mathbf{C}})$ must be stable. Because the characteristic polynomial is available, a Routh-Hurwitz stability test based on triangularization of a Hurwitz matrix is very suitable for use at each stage of the optimization. To obtain a stable $(\mathbf{F} + \mathbf{g}\mathbf{k}\hat{\mathbf{C}})$ to start the parameter optimization process after an arbitrary initial choice of sensor positions and gains, a preliminary parameter optimization is used to decrease the sum of the positive real parts of the roots until all are stable. Newton-Raphson iteration on the characteristic polynomial of the companion matrix $(\mathbf{F} + \mathbf{g}\mathbf{k}\hat{\mathbf{C}})$ is used to find the roots.

An available hill-climbing program based on the method of Rosenbrock[12] is applied in the preliminary optimization as well as in the main parameter optimization process, which commences when a stable $(\mathbf{F} + \mathbf{g}\mathbf{k}\hat{\mathbf{C}})$ has been found. The elements of the measurement matrix **C** must be recalculated after each change of sensor positions. An auxiliary constraint during optimization is that sensor positions must be within the length of the rocket. Optimization is started from different sets of initial sensor positions and gains to ensure that a global minimum is found.

About 30 search trials are required to find a stable starting point, and about 50 trials per variable will give satisfactory convergence. For three sensors, this means $6 \times 50 = 300$ Liapunov solutions are required, with an operation count of $300 \times \frac{1}{3}n^3 = 100n^3$. In addition, computation of **C** is also required and evaluation of $\mathbf{C}\mathbf{H}^{-1}$ adds a further operation count of about $\frac{1}{3}n^3$ at each stage. Typically about 30 sec of computation time is required to solve the three sensor problem from an arbitrary starting point, using the IBM 360/65.

Results

To stabilize a rocket which is open-loop unstable at least one angular position sensor and one angular rate sensor are required. In addition, if the angle of attack is to be used for control an accelerometer is needed, because the angle of attack is a component in its output. Therefore three sensors, one of each type, are assumed for the suboptimal solution.

Unique minima were always found. All state trajectories are in response to an initial condition $\phi = 0.10$ rads in pitch angle.

Figures 2 and 3 show the rigid body and flexible mode response curves for both the optimal and suboptimal controls. It is seen that the differences are very small both in the rigid body and the flexible mode responses. Flexible mode damping

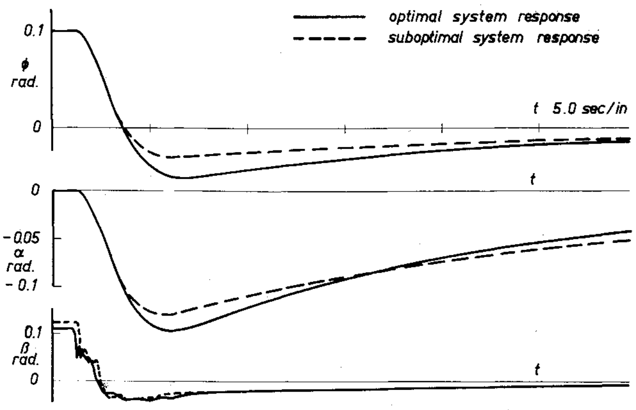

Fig. 2 Rigid body responses and engine gimbal angle.

is good, within the suboptimal case the first two modes damped somewhat better than optimal and the third less. But gimbal angle oscillation is considerable, and may not be tolerable.

The value of the cost function is 9.276×10^{12} for the suboptimal control vs 9.053×10^{12} for the optimal one. The closed loop eigenvalues also differ only little, as shown in Table 3, which shows the open loop eigenvalues of the rocket and the closed loop eigenvalues for the optimal and the suboptimal controls. Thus, response very close to optimal can be obtained using only three sensors, at computed positions on the rocket.

Table 4 shows the sensor positions and gains for the suboptimal controls. The position sensor is near the center of pressure, the rate sensor is near the tail, and the accelerometer is in a tail location where all modes have the same sign. Figure 1 serves to interpret the positions in terms of the mode shapes.

To evaluate the possible effects of contraints on sensor positions and of arbitrary choice of these positions, the suboptimal control is computed also with the position and rate sensors restricted to be within one meter of each other. This simulates the use of an accelerometer plus a rate sensor, of which the output is integrated to provide also a position signal.

Eigenvalues, gains and positions are shown in the appropriate tables. The cost is 9.640×10^{12}, so not much higher than for the three sensor case. However, the flexible mode damping is reduced and, worse, the three rigid body poles are all real, instead of having the desired configuration of a complex pair plus a real pole, which should be near the origin to minimize drift.

The best sensor positions for the suboptimal controls tend to be in three areas, namely 12–13 m, 20–22 m, and 58–66 m from the tail end of the 100-m-long rocket. The best accelerometer location appears fixed at about 20 m, a point where all mode slopes are the same.

Conclusions

By optimizing sensor positions as well as gains, response very close to optimal is possible using only three sensors. The use of the control canonical form is a very efficient basis for optimal control computations in rocket problems. For single-input systems it allows a matrix problem to be solved largely as a vector problem. In this study the transformation has led to very effective programs for solution of the matrix Riccati equation and the Liapunov equation.

Where in previous papers on suboptimal rocket control only sensor gains are adjustable, this study indicates that significant further improvement is possible by also optimizing sensor positions, and that arbitrary choice of these positions may lead to undesirable response characteristics.

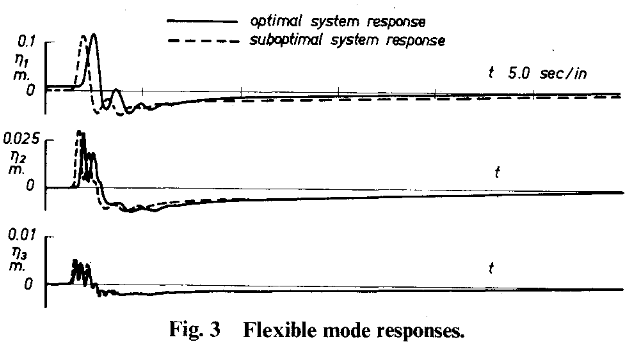

Fig. 3 Flexible mode responses.

Table 3 System eigenvalues

System	Eigenvalues	
Open loop system	0.046	$-0.027 \pm 5.46\,i$
	0.433	$-0.065 \pm 12.99\,i$
	-0.493	$-0.091 \pm 18.28\,i$
Optimal control	-0.0556	$-2.714 \pm 12.79\,i$
	$-0.717 \pm 0.514\,i$	$-1.758 \pm 18.06\,i$
	$-0.794 \pm 5.41\,i$	
Suboptimal control	-0.039	$-3.181 \pm 13.09\,i$
	$-0.829 \pm 0.607\,i$	$-1.235 \pm 17.96\,i$
	$-1.00 \pm 5.22\,i$	
Suboptimal control with constraint	-0.057	$-1.00 \pm 5.40\,i$
	-0.495	$-3.31 \pm 13.23\,i$
	-1.121	$-1.17 \pm 17.85\,i$

Table 4 Sensor positions and feedback gains

		Three sensors	Three sensors with constraint
Position sensor	Position:	58.136	12.245
	Gain:	-1.168	-0.8237
Angular rate sensor	Position:	12.550	13.091
	Gain:	-1.530	-1.5788
Accelerometer	Position:	20.453	20.164
	Gain:	0.00995	0.00911

References

[1] Rynaski, E. and Whitbeck, R., "Optimal Control of a Flexible Launch Vehicle," Rept. IH-2089-F-1, July 1966, Cornell Aeronautical Lab., Buffalo, N.Y.

[2] Fisher, E. E., "An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Launch Vehicle," *AIAA Journal*, Vol. 3, No. 7, July 1965, pp. 1262–1267.

[3] Kosut, R. L., "Suboptimal Control of Linear Time-Invariant Systems Subject to Control Structure Contraints," *IEEE Transactions onAutomatic Control*, Vol. AC-15, No. 5, Oct. 1970, pp. 557–563.

[4] Athans, M. and Levine, W. S., "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems," *IEEE Transactions on Automatic Control*, Vol. AC-15, No. 1, Feb. 1970, pp. 44–50.

[5] Axsäter, S., "Suboptimal Time-Variable Feedback Control of Linear Dynamic Systems with Random Inputs," *International Journal of Control*, Vol. 4, No. 6, 1969, pp. 549–566.

[6] Davison, E. J. and Rau, N. S., "The Optimal Output Feedback Control of a Synchronous Machine," IEEE Power Systems Paper 71 TP 102-PWR, 1971.

[7] Kleinman, D. L., "On an Iterative Technique for Riccati Equation Computations," *IEEE Transactions on Automatic Control*, Vol. AC-13, No. 4, Feb. 1968, pp. 114–115.

[8] Davison, E. J. and Man, F. T., "The Numerical Solution of $A'Q+QA = -C$," *IEEE Transactions on Automatic Control*, Vol. AC-13, No. 4, Aug. 1968, pp. 448–449.

[9] Wilkinson, J. H., *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, England, 1965, p. 369.

[10] Kalman, R. E. and Arbib, M. A., "Coordinate Changes and Canonical Forms," Chap. 2.4 *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969, pp. 41–46.

[11] Molinari, B. P., "Algebraic Solution of Matrix Linear Equations in Control Theory," *Proceedings of the IEE*, Vol. 116, No. 10, Oct. 1969, pp. 1748–1754.

[12] Rosenbrock, H. H., "An Automatic Method of Finding the Greatest or Least Value of a Function," *Computer Journal*, Vol. 3, No. 3, Oct. 1960, pp. 175–184.